ON THE STABILITY OF MOTION OF A RIGID BODY CONTAINING A FLUID POSSESSING SURFACE TENSION

(OB USTOICHIVOSTI DVIZHENIIA TVERDOGO TELA S ZHIDKOST'IU, OBLADAIUSHCHEI POVERKHNOSTNYM NATIAZHENIEM)

PMM Vol.28, № 4, 1964, pp. 746-753

V.V.RUMIANTSEV
(Moscow)

(Received May 6, 1964)

In the paper [1] a theorem was established which reduced the question of the stability of stationary motion (in particular, equilibrium) of a rigid body with a cavity wholly or partially filled with an ideal or viscous fluid to the problem of the minimum of the potential energy. The surface tension of the fluid was not taken into account in the paper. However, in a number of cases, particularly under zero-gravity conditions, the inclusion of surface tension may prove to be significant [2]. Theorems on the stability of equilibrium and stationary motion of a rigid body containing a cavity filled with a fluid possessing surface tension are proved below.

1. Let us imagine a rigid body having a simply connected cavity to be constrained by some stationary frictionless connections, or to be free. We denote by q_j $(j = 1, 2, \ldots, n;\ n \leqslant 6)$, the Lagrangian coordinates defining the position of the rigid body in a fixed system of coordinates $O_1\xi\eta\zeta$. Let there be acting on the body given potential forces possessing a force function, which we write as $U(q_1, \ldots, q_n)$. In addition to the fixed coordinate system, we will consider also a moving system of coordinate axes $Oxyz$, rigidly fixed to the rigid body. We assume that the cavity of the body is completely filled with two homogeneous, incompressible, ideal, immiscible fluids 1 and 2, which have surface tension and are subjected to body forces with potential $U_i(\xi, \eta, \zeta)$.

We denote the density by ρ_i, the pressure by p_i, the volume by τ_i, and the area of the surface of each of the fluids by S_i $(i = 1, 2)$. Generally speaking, $S_i = S_{12} + \sigma_{i3}$, where S_{12} is the area of the dividing surface of the fluids, and σ_{i3} is the area of the surface of the walls of the cavity wetted by the ith fluid. We denote the line of intersection of the dividing surface with the walls of the cavity by σ, and henceforth we shall assume for simplicity that in the nighborhood of this line the surface of the walls of the cavity have no sharp edges. However, the case may arise where one of the fluids is completely surrounded by the other fluid and does not come into contact with the walls of the cavity. In this case the line σ does not exist, and the area of the surface of the inner fluid is $S_1 = S_{12}$.

We note that the problem as formulated includes the case where the cavity is partially filled by a homogeneous incompressible fluid. In this case the free surface of the fluid S_{12} borders either on air, in which the pressure p_2 is constant, or on vacuum, where the pressure p_2 is taken to be zero.

Following Gauss, we assume that due to the contact of two different media along a certain surface there will be tensile forces having a potential equal to the product of the area of the surface of contact and the coefficient of surface tension α_{ir} $(i = 1,2;\ r = 1,2,3)$ [3] which depends on the nature of both media. Obviously $\alpha_{ir} = \alpha_{ri}$.

As is well known [4], the form of the differential equations of motion of the fluid is independent of the presence or absence of the surface tension forces, which however affect the form of the boundary conditions on the surface dividing the fluids. The boundary conditions, as well as the equations of motion of the system, may be derived from the principle of least action in the Hamilton-Ostrogradskii form [5]. According to this principle, for any possible motion of the system

$$\int_{t_0}^{t_1}\left[\delta T + \sum_{\nu}(X_\nu\delta\xi_\nu + Y_\nu\delta\eta_\nu + Z_\nu\delta\zeta_\nu)\right]dt = 0 \tag{1.1}$$

where T is the kinetic energy of the system, X_ν, Y_ν, Z_ν are the projections of the active forces on the fixed axes, δ is the variational symbol (for the change in a possible displacement) for the corresponding quantity, and

$$\delta\xi_\nu = \delta\eta_\nu = \delta\zeta_\nu = 0 \quad \text{for } t = t_0,\ t = t_1$$

With the above assumptions regarding the forces acting on the system, the total work done by these forces in a virtual displacement becomes

$$\sum_{\nu}(X_\nu\delta\xi_\nu + Y_\nu\delta\eta_\nu + Z_\nu\delta\zeta_\nu) = -\delta V - \alpha_{12}\delta S_{12} - \alpha_{13}\delta\sigma_{13} - \alpha_{23}\delta\sigma_{23}$$

where

$$V = -U - \rho_1\int_{\tau_1} U_1 d\tau_1 - \rho_2\int_{\tau_2} U_1 d\tau_2$$

denotes the potential energy of the external forces acting on the system.

It is not difficult to show that [3]

$$\delta S_{12} = \int_{S_{12}}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\delta\varkappa ds + \int_{\sigma}\delta\varkappa_1 d\sigma, \qquad \delta\sigma_{13} = -\delta\sigma_{23} = \int_{\sigma}\delta\varkappa_2\, d\sigma \tag{1.2}$$

Here R_1 and R_2 denote the principal radii of curvature of the surface S_1 at a given point, and are considered positive if the corresponding center of curvature lies on the same side of the surface as the fluid 1, and negative in the opposite case;

$$\delta\varkappa = \delta\mathbf{r}\cdot\mathbf{n}, \qquad \delta\varkappa_1 = \delta\mathbf{r}\cdot\mathbf{n}_1, \qquad \delta\varkappa_2 = \delta\mathbf{r}\cdot\mathbf{n}_2$$

where $\delta\mathbf{r}$ is a possible displacement of a point of the surface S_1 or the line σ relative to the rigid body, $\mathbf{n}$ is the unit outer normal to the surface S_1, $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit outer normals to the contour σ of the surfaces S_{12} and σ_{13}, lying in the respective tangent planes to these surfaces. It is obvious that for the portion S_{12} of the surface S_2 the vector $\mathbf{n}$ is the inner normal. On the surface σ_{13}, $\delta\varkappa = 0$ owing to the impenetrability of the rigid wall. We denote the angle between the normals $\mathbf{n}_1$ and $\mathbf{n}_2$ by θ. It is easily seen that

$$\delta\varkappa_1 = \delta\varkappa_2 \cos\theta \tag{1.3}$$

Taking into account the continuity conditions for the fluid and introducing the undetermined multipliers $p_i(x, y, z, t)$ which define the hydrodynamic pressure, we obtain from (1.1) the equations of motion of the system [5] (which are not set down here because they are not required in the subsequent analysis), as well as Equation

$$\int_{t_0}^{t_1}\left\{\iint_{S_{12}}\left[p_1 - p_2 - \alpha_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]\delta\varkappa\, ds - \int_{\sigma}(\alpha_{12}\cos\theta + \alpha_{13} - \alpha_{23})\,\delta\varkappa_2\, d\sigma\right\} dt = 0$$

Owing to the arbitrariness of the possible displacements of a fluid particle (arbitrary to the extent that they satisfy the continuity equation), we obtain Laplace's formula [4] for the pressure on the dividing surface of the fluids

$$p_1 - p_2 = \alpha_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{1.4}$$

and also the formula for the edge angle θ

$$\cos\theta = (\alpha_{23} - \alpha_{13}) / \alpha_{12} \tag{1.5}$$

which must be satisfied at an arbitrary instant of time. We note that Formula (1.5) has exactly the same form as the formula for the edge angle for a fluid in equilibrium, which is usually obtained either from the principle of virtual displacements [3], or from the condition of equilibrium of the three surface tension forces [4]. Consequently, the moving surface dividing the two dissimilar fluids forms the same angle with the rigid wall as in equilibrium, if of course the quantities α_{ir} are the same constants.

For air the coefficient α_{23} is usually taken to be zero, and Formula (1.5) assumes the form

$$\cos\theta = -\alpha_{13} / \alpha_{12} \tag{1.6}$$

Furthermore, we will consider only the cases where the motion of the rigid body is continuous, while the motion of the fluid is accomplished in a smooth manner, so that the coordinates of a fluid particle are continuous functions of their initial values and time.

In view of the assumptions that the fluid is ideal and that the constraints imposed upon the rigid body are stationary, according to the theorem regarding the kinetic energy of the system we have

$$dT = -dV - \alpha_{12}dS_{12} - \alpha_{13}d\sigma_{13} - \alpha_{23}d\sigma_{23}$$

whence we obtain the energy integral

$$T + V + \alpha_{12}S_{12} + \alpha_{13}\sigma_{13} + \alpha_{23}\sigma_{23} = \text{const} \tag{1.7}$$

N o t e . The integral (1.7) may also be obtained from the equations of motion of the system, and the following equation may be derived by the usual method [5]:

$$\frac{d}{dt}(T + V) = -\int_{S_{12}} (p_1 - p_2)\, u_n\, dS$$

where u_n denotes the projection of the relative velocity of the fluid on the normal n to the surface S_{12}. Using Formulas (1.2) to (1.5), we immediately obtain from this equation the integral (1.7).

Thus under the assumed conditions the total mechanical energy of the system, consisting of the kinetic energy T of the rigid body and fluid, the potential energy V of the external forces applied to the system, and the surface energy $\alpha_{12}S_{12} + \alpha_{13}\sigma_{13} + \alpha_{23}\sigma_{23}$, of the fluid, remains constant throughout the motion.

2. We denote the potential energy of the system by $F = V + \alpha_{12}S_{12} + \alpha_{13}\sigma_{13} + \alpha_{23}\sigma_{23}$. According to the principle of virtual displacements Equation

$$\delta F = 0 \tag{2.1}$$

is the condition of equilibrium of the rigid body with fluid in its cavity. Transforming from the absolute coordinates ξ, η and ζ of a fluid particle to the relative coordinates x, y and z, we write the potential function of the body forces in the form $U_1(x, y, z, q_j)$, retaining the previous notation.

Writing Equation (2.1) in an explicit form, we have

$$\sum_{j=1}^{n} \frac{\partial V}{\partial q_j}\delta q_j - \sum_{i=1}^{2} \rho_i \int_{\tau_i} \left(\frac{\partial U_1}{\partial x}\delta x + \frac{\partial U_1}{\partial y}\delta y + \frac{\partial U_1}{\partial z}\delta z\right) d\tau_i + \\ + \alpha_{12}\int_{S_{12}} \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\delta\varkappa\, ds = 0$$

In view of the independence of δq_j and δx, δy, δz, we immediately obtain from this equation the equation of equilibrium of the rigid body and fluid

$$\frac{\partial V}{\partial q_j} = 0 \qquad (j = 1, \ldots, n) \tag{2.2}$$

The remaining part of the equality leads to the equation of the dividing surface S_{12} of the fluids in equilibrium. If we use Green's formula and take into account the incompressibility condition, we obtain

$$\sum_{i=1}^{2} \rho_i \int_{\tau_i} \left(\frac{\partial U_1}{\partial x}\delta x + \frac{\partial U_1}{\partial y}\delta y + \frac{\partial U_1}{\partial z}\delta z\right) d\tau_i - \alpha_{12}\int_{S_{12}} \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\delta\varkappa\, ds = \\ = \int_{S_{12}} \left[(\rho_1 - \rho_2)\, U_1 - \alpha_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]\delta\varkappa\, ds = 0$$

Conservation of volume of the fluid requires that the function $\delta\varkappa$ satisfy the condition

$$\int_{S_{12}} \delta\varkappa \, ds = 0$$

while it is arbitrary elsewhere; hence we obtain from the latter equality the equation of the dividing surface of the fluids in equilibrium

$$(\rho_1 - \rho_2) U_1 - \alpha_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \text{const} \tag{2.3}$$

We shall now consider the case where the constraints applied to the rigid body allow rotation of the whole system as a single rigid body around some fixed straight line, for example, the ζ-axis, and the forces acting on the system exert no moment about this line. In this case the kinetic and potential energy of the system clearly do not depend on the angle of rotation q_n of the body about the ζ-axis, and the equations of motion have the integral [5]

$$G_\zeta = k = \text{const} \tag{2.4}$$

where G_ζ is the component of the angular momentum of the system along the ζ-axis.

We also introduce the coordinate system $O_1\xi_1\eta_1\zeta$ which rotates about the ζ-axis with angular velocity ω. If the value of ω is chosen such that at an arbitrary instant of time the relation

$$\omega J = k \tag{2.5}$$

is satisfied, then the energy integral (1.7) may be written in the form [1]

$$T_* + \frac{1}{2}\frac{k^2}{J} + V + \alpha_{12}S_{12} + \alpha_{13}\sigma_{13} + \alpha_{23}\sigma_{23} = \text{const} \tag{2.6}$$

Here T_* denotes the kinetic energy of the system in its motion relative to the coordinate axes $O_1\xi_1\eta_1\zeta$, while J is the moment of inertia of the system about the ζ-axis.

We introduce the notation

$$W = \frac{1}{2}\frac{k_0^2}{J} + V + \alpha_{12}S_{12} + \alpha_{13}\sigma_{13} + \alpha_{23}\sigma_{23} \tag{2.7}$$

for the change in potential energy of the system.

From the d'Alambert-Lagrange principle [1], it follows that the Equation

$$\delta W = 0 \tag{2.8}$$

is the condition for the establishment of the motion in which the whole system rotates as a single rigid body about the ζ-axis with angular velocity $\omega_0 = k_0/J_0$, where k_0 and J_0 are the values of the constant k and the moment of inertia J for which such a motion is set up.

From condition (2.8) we easily obtain Equations

$$\frac{\partial W}{\partial q_j} = -\frac{1}{2}\omega_0^2\frac{\partial J}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 \qquad (j = 1, \ldots, n-1) \tag{2.9}$$

for the coordinates q_i of the rigid body in its stationary motion, as well as the equation of the dividing surface of the fluids in this motion

$$(\rho_1 - \rho_2)\left[U_1 + \frac{1}{2}\omega_0^2(\xi^2 + \eta^2)\right] - \alpha_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \text{const} \qquad (2.10)$$

For $\omega_0 = 0$, Equation (2.10) takes the form of Equation (2.3).

The constants appearing on the right-hand sides of Equation (2.3) or (2.10) are determined from the known values of the radii of curvature R_i and force function U_1 at any point of the dividing surface [3].

The formula for the average curvature $1/R_1 + 1/R_2$ of a surface determined from the coefficients of the first and second quadratic forms of Gauss [6] is well-known in differential geometry). Equations (2.3) and (2.10) are thus the differential equations of the dividing surface of the fluids in equilibrium or in stationary motion, the shape of which is determined by integrating these equations. The integrals of Equations (2.3) and (2.10) must satisfy boundary conditions of the form of (1.5) or (1.6) if the dividing surface intersects the walls of the cavity. Experiment shows that the angle θ may be either an acute or right or obtuse angle, depending on the nature of the contiguous media. For a liquid-air surface the angle θ is obtuse for a "nonwetting" wall, and $\alpha_{13} > 0$, while for a "wetting" wall θ is acute, and in this case $\alpha_{13} < 0$.

If $|\alpha_{13}/\alpha_{12}| > 1$, then a line of intersection of the free surface of the fluid with the walls of the cavity does not exist, and the fluid is distributed over the whole surface of the walls of the cavity.

Far from the walls of the cavity, the shape of the dividing surface of the fluids in equilibrium depends on the relations between the magnitudes of the surface tensions and the body forces acting on the fluids. Thus, for example, in a gravitational field the shape of the surface is determined [4] by the capillarity constant $a = \sqrt{2\alpha_{12}/g\varrho}$.

3. We now consider the question of the stability of equilibrium or stationary motion of a rigid body with a fluid in its cavity. We shall agree that by the stability of the present system with an infinite number of degrees of freedom, we mean stability with respect to the coordinates q_j and velocities q_j' of the rigid body, the kinetic energy of the fluid $T_*^{(2)}$, and the distance l of the dividing surface from the equilibrium surface (or the displacement ∇ of the shape of the fluid from the equilibrium shape). In addition to conditions (2.1) of [1], we require that the inclination of the initial perturbation in the dividing surface to the equilibrium surface be sufficiently small at each point of its surface. We shall assume that the equilibrium surface is connected, the direction of its normal varying continuously for continuous variations of a point on the surface, and that the curvatures of its principal normal sections are everywhere finite. We shall always choose the distance l to be smaller than the least of all the radii of curvature of the normal sections of the equilibrium surface. Under these conditions the following theorems hold.

T h e o r e m 3.1. If Expression

$$F = V + \alpha_{12}S_{12} + \alpha_{13}\sigma_{13} + \alpha_{23}\sigma_{23}$$

has an isolated minimum F_0 for the equilibrium position of the rigid body and fluid in its cavity, then the equilibrium position is stable.

T h e o r e m 3.2 . If in the state of stationary motion of the rigid body with fluid in its cavity Expression

$$W = \frac{1}{2}\frac{k_0^2}{J} + V + \alpha_{12}S_{12} + \alpha_{13}\sigma_{13} + \alpha_{23}\sigma_{23}$$

has an isolated minimum W_0, then the steady motion is stable.

The minimum of Expression F or W is taken to have the same meaning as in [1], i.e. either with respect to q_j, l (for $\nabla > \varepsilon l$), or with respect to q_j, ∇.

P r o o f . We perturb the system from its state of stationary motion, imparting to its points certain sufficiently small initial displacements and velocities. Left to itself, the system will continue to move in accordance with the energy integral (2.6), which may be rewritten in the form

$$T_* + W + \frac{1}{2}\frac{k^2 - k_0^2}{J} = T_*^{(0)} + W^{(0)} + \frac{1}{2}\frac{k^2 - k_0^2}{J^{(0)}} \tag{3.1}$$

where (0) denotes the initial value of the corresponding quantity, and k is the value of the integral constant for the perturbed motion.

Let A be some arbitrarily small positive number which does not exceed a given number L defining the region of stability, which we will in any case assume to be less than the number E which defines the region of minimum W [1]. We denote by W_1 the smallest possible value that can be assumed by the the expression W if the absolute value of the distance l or one of the coordinates q_j $(j = 1, \ldots, n-1)$ is equal to A, while the rest of these quantities and the displacement ∇ satisfy conditions $|q_j| \leqslant A$, $|l| \leqslant A$, $\nabla \geqslant \varepsilon l$, where clearly $W_1 > W_0$. We choose the number A to be so small that the inequality $|W_1 - W_0| < L$ will be satisfied.

We choose the initial positions and velocities of points of the system such that the inequality

$$T_*^{(0)} + W^{(0)} + \frac{1}{2}(k^2 - k_0^2)\left(\frac{1}{J^{(0)}} - \frac{1}{J}\right) < W_1$$

will be fulfilled for all values which J may assume satisfying conditions

$$|q_j| \leqslant A, \quad |l| \leqslant A, \quad \nabla \geqslant \varepsilon l \tag{3.2}$$

For such a choice of initial conditions, at all subsequent times of the motion for which the inequalities (3.2) are satisfied, we will have in accordance with the energy integral (3.1) the inequality

$$T_* + W < W_1 \tag{3.3}$$

from which it follows that $W < W_1$. This inequality will be satisfied at least until $|q_j|$ and $|l|$ exceed the number A. But the initial values of $|q_j|$ and $|l|$ were chosen to be less than A, and the initial displacement $\nabla > \varepsilon l$ and since q_j, l, ∇ vary continuously with time, then $|q_j|$ and $|l|$ can neither exceed A nor equal A before that time.

However, the equalities

$$|q_j| = A, \qquad |l| = A$$

are impossible under the condition $\nabla \geqslant \varepsilon l$ in view of the inequality (3.3). Consequently, if the motion of the system is continuous, so that q_j, l, ∇ vary continuously with time, then starting at the initial instant of time, we have the inequalities

$$|q_j| < L, \quad |q_j'| < L, \quad |l| < L, \quad |T_*^{(2)}| < L, \quad \nabla \geqslant \varepsilon l$$

all of which will continue to be satisfied as long as the last one of them is observed. Thus Teorem 3.2 is proved. The validity of Theorem 3.1 follows from the proof.

We note that a theorem similar to Theorem 3.1 can also be stablished for the case of relative equilibrium of a rigid body with a cavity filled with a fluid possessing surface tension if in addition to the potential forces, there acts the moment of other forces which are directed along the ζ-axis, in which case the angular velocity ω of rotation of the body about the ζ-axis remains constant throughout the motion. Under these conditions the following theorem [1] is valid.

T h e o r e m 3.3 If in the position of relative equilibrium of the rigid body and the fluid in its cavity Expression

$$W_* = V + \alpha_{12}S_{12} + \alpha_{13}\sigma_{13} + \alpha_{23}\sigma_{23} - {}^1/_2\omega^2 J$$

has an isolated minimum, then the position of relative equilibrium is stable.

Since condition $\delta W_* = 0$ for ω = const is equivalent to condition $\delta W = 0$ for k_0 = const , it is easily seen from a straightforward argument using the equality $\omega J_0 = k_0$ that the position of relative equilibrium may be compared with the stationary motion of the system. It is not difficult to see [1] that if the expression W_* has a minimum for a certain position of relative equilibrium, then the expression W for the corresponding stationary motion also has a minimum.

It was assumed above that the fluid was ideal, but Theorems 3.1 to 3.3 remain valid for a viscous liquid as well.

In the case of a viscous fluid the Euler equations are replaced by the Navier-Stokes equations; the velocity of a fluid particle in contact with the walls of the cavity is taken to be equal to the velocity of the corresponding point on the wall, and the dynamic condition (1.4) on the free surface is replaced by condition [4]

$$(p_1 - p_2)\, n_i = (\sigma_{ik}'^{(1)} - \sigma_{ik}'^{(2)})\, n_k + \alpha_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) n_i \tag{3.4}$$

where $\sigma_{ik}'^{(1)}$ and $\sigma_{ik}^{(2)}$ denote the "viscous" stress tensors. Under these conditions the equations of motion of the system reduce to the equations for the rate of energy dissipation

$$\frac{d}{dt}(T + V + \alpha_{12}S_{12} + \alpha_{13}\sigma_{13} + \alpha_{23}\sigma_{23}) = -\mu_1 \int_{\tau_1} \Phi_1 d\tau_1 - \mu_2 \int_{\tau_2} \Phi_2 d\tau_2$$

where Φ_i is defined by the Navier-Stokes formula [1], and μ_i is the coefficient of viscosity. Hence for a viscous fluid, in place of the energy integral (1.7) we have the inequality

$$T + V + \alpha_{12}S_{12} + \alpha_{13}\sigma_{13} + \alpha_{23}\sigma_{23} \leqslant T^{(0)} + V^{(0)} + \alpha_{12}S_{12}^{(0)} + \alpha_{13}\sigma_{13}^{(0)} + \alpha_{23}\sigma_{23}^{(0)} \tag{3.5}$$

and nothing is changed in the proof of Theorems given above.

In a similar manner one may validate Theorems 3.1 and 3.2 of [1] for a viscous liquid possessing surface tension, taking into account the definition (2.7).

The author thanks N.N. Krasovskii, N.N. Moiseev and G.K. Pozharitskii for discussions of the present paper.

BIBLIOGRAPHY

1. Rumiantsev, V.V., Ob ustoichivosti ustanovivshikhsia dvizhenii tverdykh tel s polostiami, napolnennymi zhidkost'iu. (On the stability of stationary motions of rigid bodies with cavities containing fluid). *PMM* Vol.26, № 6, 1962.
2. Moisseyev, N.N., Sur certains problèmes mathématiques du mouvement relatif des satellites. Dynamics of satellites. Symposium, Paris, 1962.
3. Kirchhoff, G., Mechanics (Russian translation). Izd.Akad.Nauk SSSR,1962.
4. Landau, L.D. and Lifshits, E.M., Mekhanika sploshnykh sred (Mechanics of Continuous Media). Gostekhizdat, 1953.
5. Rumiantsev, V.V., Uravneniia dvizheniia tverdogo tela, imeiushchego polosti, ne polnost'iu napolnennye zhidkost'iu (Equations of motion of a rigid body having cavities not completely filled with a fluid). *PMM* Vol.18, № 6, 1954.
6. Smirnov, V.I., Kurs vysshei matamatiki (Course of Higher Mathematics), Vol.II, Gostekhizdat, 1952.
7. Pozharitskii, G.K. and Rumiantsev, V.V., Zadacha minimuma v voprose ob ustoichivosti dvizheniia tverdogo tela s polost'iu, zapolnennoi zhidkost'iu (The problem of the minimum in the question of stability of motion of a solid body with a liquid-filled cavity). *PMM* Vol.27, № 1, 1963.

Translated by F.A.L.